Available online at www.sciencedirect.com
ScienceDirect
Physica A 387 (2008) 425–431
PHYSICA A
www.elsevier.com/locate/physa

Analysis of high dimensional non-hyperbolic coupled systems through finite-time Lyapunov exponents

Marcelo M. Disconzi, Leonardo G. Brunnet*

Instituto de Física, Universidade Federal do Rio Grande do Sul, Caixa Postal 15051, 91501-970, Porto Alegre, RS, Brazil

Received 18 September 2007
Available online 29 September 2007

Abstract

In this work, we investigate the relations among the Unstable Dimension Variability (UDV) and phase space dimensions in a coupled map lattice with diffusive coupling. Studying a simple system with UDV at low dimensions, we give theoretical support and numerical evidence to the statement that, from some fixed dimensional value onwards, there is no UDV. More precisely, we construct a high dimensional non-hyperbolic system without UDV.

Keywords: Chaotic systems; Shadowing; Unstable dimension variability; Finite time Lyapunov exponents

1. Introduction

It is a well known fact that chaotic systems present the so-called *sensitivity to initial conditions*: trajectories starting nearby in phase-space diverge fast with time, and small errors at the beginning yield huge differences after some iterations. As in a large number of problems of physical interest, direct analytical methods are not available; we are often obliged to use numerical simulations to study them. Since numerical algorithms always present errors due to any computer's finite precision, a numerical trajectory starting *exactly* at a given point will eventually diverge, apart from a fiducial trajectory (without round off errors).

Although the limits imposed by computers prevent us from following a trajectory for long periods of time, properties of physical interest are usually obtained from *ensembles* of trajectories. Therefore, if it can be ensured that some trajectory stays close to the numerical one – even if their initial conditions differ – all the relevant data will be reliable. A trajectory that presents an error of size δ at each step is called a δ*-pseudo-trajectory* (we usually think of δ-pseudo-trajectories as those generated by a computer simulation; in this case δ can be interpreted as the round off error). When the system exhibits the property of having its δ-pseudo-trajectories close to exact solutions, we say that it is (or that its trajectories are) *shadowable*. Formally, for a given map $f(x)$, a sequence $\{x_n\}_{n=0}^{\infty}$ is a δ-pseudo-trajectory when $d(x_{n+1}, f(x_n)) < \delta,\ n = 0, 1 \ldots$, and a δ-pseudo-trajectory is η-shadowable if there exists a point y such that $d(x_n, f^n(y)) < \eta,\ n = 0, 1 \ldots$. Here $d(u, v)$ denotes the distance between two points u and v.

* Corresponding author.
E-mail addresses: disconzi@if.ufrgs.br (M.M. Disconzi), leon@if.ufrgs.br (L.G. Brunnet).
URL: http://pcleon.if.ufrgs.br (L.G. Brunnet).

0378-4371/$ - see front matter
doi:10.1016/j.physa.2007.09.012

In most applications, our space is simply $\mathbb{R}^n$, and in this case the distance is given by the standard Euclidean norm: $d(u, v) = \|u - v\|$. We say that the dynamical system itself is shadowable if for every η there exists a δ such that every δ-pseudo-trajectory is η-shadowable.

Hyperbolic systems with local product structure are always shadowable [8], although shadowability is not a feature found uniquely in hyperbolic families (for an example, see [5]). If the system is not shadowable, we are not able to assure numerical data faithfulness and numerical procedures may yield incorrect outcomes.

This lack of shadowability seems to be found in a large number of models which present a phenomenon known as *Unstable Dimension Variability* (UDV): two hyperbolic fixed points embedded in a chaotic attractor present different numbers of unstable dimensions, so that the number of unstable directions is not constant along a dense trajectory.

We remark, however, that the exact connections between UDV and the lack of shadowability are far from being completely understood: if on one hand Yuan and Yorke [19] have shown that UDV is a sufficient condition to the non-existence of shadowability in the great class of diffeomorphisms which fall into their hypothesis, on the other hand we do not have a *formal proof* which states that the absence of UDV implies shadowability. UDV is also found and prevents shadowing properties in robustly transitive three dimensional partially hyperbolic diffeomorphisms [4]. For the more general class of transformations that are not diffeomorphisms, even a sufficient condition for the absence of shadowability is unknown, although we have strong evidence that in such cases UDV also implies no shadowability [15]. Because of thist, much work has been done to identify the presence of UDV in an attempt to get information about the shadowing properties of the systems. Such intense work has revealed that UDV seems to be a typical phenomenon of high dimensional non-hyperbolic systems [10].

Although the intense work of Grebogi and collaborators (there are many references, see for example [9,15,3,11,16,17,13,14,18,10]) has been of fundamental importance in showing a large number of systems with UDV, we assert that this is not a universal property of non-hyperbolic high dimensional systems. In fact, the aim of this work is the construction of a high-dimensional non-hyperbolic system without UDV. More precisely, we construct a coupled map system in which the expected characteristics of UDV systems – such as oscillations around zero of the finite-time Lyapunov exponents distribution – are present at low phase space dimensions. However, increasing the dimension, we find a critical dimension above which there is no UDV. This is the point we want to stress in this work: the dependence of UDV on the phase space dimension.

The finite-time Lyapunov exponent is the tool used to identify UDV [2,6]. We stress that there is no proof showing that fluctuation of the finite-time Lyapunov exponent implies UDV. However, it has been conjectured that the fluctuation of this exponent is a typical UDV fingerprint [15], and therefore such a method has been widely used.

2. Construction of the system

The local transformation is given by:

$$f(x) = \begin{cases} \dfrac{2}{1-L}x & 0 \le x \le \dfrac{1-L}{2} \\ 1 & \dfrac{1-L}{2} < x < \dfrac{1+L}{2} \\ \dfrac{2}{1-L}(1-x) & \dfrac{1+L}{2} \le x \le 1 \end{cases} \tag{1}$$

where L is a parameter that defines the plateau length (see Fig. 1).

Once a trajectory reaches the region $\mathcal{C} = (\frac{1-L}{2}, \frac{1+L}{2})$, it goes to 1 and then to zero. As the dynamics out of $\mathcal{C}$ are expansive, we have that $x = 0$ is a global attractor to the system. The point $x = 0$ is also a linearly unstable fixed point, which makes every δ-pseudo-orbit go away from zero after passing by its neighborhood, no matter how small δ is. As a consequence, the system is not shadowable: any orbit will necessarily pass by the region $\mathcal{C}$ and go to zero; a δ-pseudo-orbit which follows it will not (due to the error δ) reach $x = 0$, and then it will go apart from the true one. Because of that we call $\mathcal{C}$ the *critical region*.

Notice that: (i) the breakdown of shadowability occurs when the orbit visits the region $\mathcal{C}$; and (ii) the parameter L determines for how long an orbit stays out of $\mathcal{C}$: since the dynamics out of $\mathcal{C}$ are chaotic, we can make the analogy of f with a random variable in the [0, 1] interval. The probability of $\mathcal{C}$ to be reached is proportional to L and the average time it takes to do so (starting from random initial conditions) would be proportional to $1/L$. Indeed, computing the

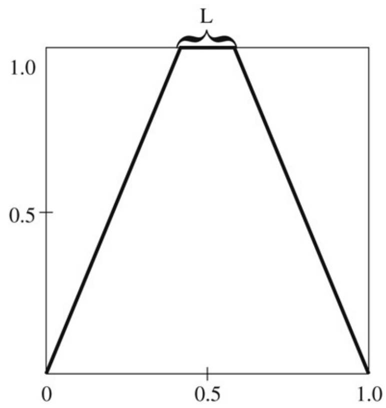


Fig. 1. The local map.

number of iterations the system takes to enter the region $\mathcal{C}$ for different values of L, we find out that the *shadowing time* T is proportional to $1/L$. Notice that in the limit $L \to 0$, we have the so-called tent-map, which is shadowable [5].

The situation is quite different when we couple the system. Consider N local maps coupled by means of a one-dimensional diffusive coupling:

$$x_i^{t+1} = (1-\epsilon) f(x_i^t) + \frac{\epsilon}{2}(f(x_{i-1}^t) + f(x_{i+1}^t)) \tag{2}$$

$i = 1, \ldots, N$ with periodic boundary conditions. ϵ is the coupling parameter.

Let us consider a configuration where $x_i^t \in \mathcal{C}$ and $x_{i-1}^t \notin \mathcal{C}$ for some i. Suppose also that ϵ is small but non-null. Then, by the definition of the local dynamics $f(x_i^t) = 1$ and $0 \le f(x_{i-1}^t) < 1$, which implies

$$1 - \epsilon + \frac{\epsilon}{2} f(x_{i+1}^t) \le x_i^{t+1} < (1-\epsilon) + \frac{\epsilon}{2}(1 + f(x_{i+1}^t)). \tag{3}$$

So, as $0 \le f(x_{i+1}^t) \le 1$, we can put upper and lower bounds to $f(x_1^{t+1})$ as $f(x_i^{t+1}) \in [1-\epsilon, 1)$. Now, f maps $[1-\epsilon, 1)$ in the neighborhood of, but is different from, zero. The hypothesis $\epsilon \ll 1$ implies the next iterate of each site will be strongly driven by the local dynamics (see Fig. 2). As 0 is an unstable fixed point to the local dynamics, the orbit of the site i will escape from zero. Therefore, the orbit of x_i is no longer attracted to zero when the coupling is turned on. This would not be so only if we had $x_{i-1}^t, x_i^t, x_{i+1}^t \in \mathcal{C}$. As i is an arbitrary site in the lattice, we infer that in the coupled case the critical region is $\mathcal{C}^N = \mathcal{C} \times \cdots \times \mathcal{C}$ (N times).

From the previous arguments, we conclude that the mechanism by which the real orbit stays at zero, whereas the pseudo-orbit goes away from it, can be avoided if we can assure that the system does not enter into the critical region $\mathcal{C}^N$. To do that, take L so small and N sufficiently large such that the volume of $\mathcal{C}^N$ in the phase space becomes irrelevant when compared to $1^N - \mathcal{C}^N$. As the probability of finding an orbit in $\mathcal{C}^N$ is proportional to its volume, we have the desired result.

We remark that our local dynamics are not hyperbolic. Such an absence of hyperbolicity persists when we couple the system, because we are dealing with very small ϵ values (see Fig. 2). We shall return to this point at the end of the paper.

We recall that the relations between UDV and shadowability are not well established. However, our arguments point in the direction of a high-dimensional non-hyperbolic system, where the mechanism of shadowing breakdown seems to be avoided. Considering the great amount of evidence relating UDV and the lack of shadowability (see introduction), we should expect to not find UDV in our system. In the next section, we reinforce our geometrical arguments with numerical evidence, and also use such numerical data to determine how small and how large must L and N be.

3. Computing the finite-time Lyapunov exponent

In this section, we develop the numerical part of the work. We use the finite-time Lyapunov exponent distribution (or distribution for short) as a tool to identify UDV. We notice that we have as many distributions as the number

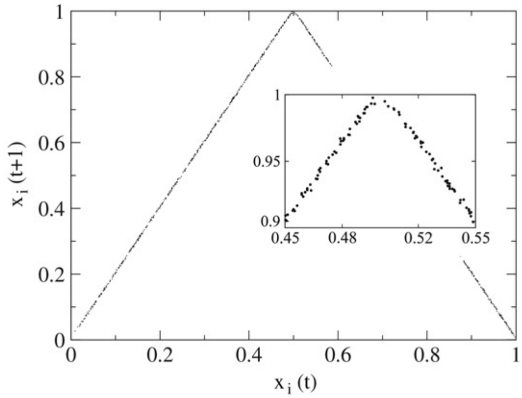


Fig. 2. First return map of a site chosen at random in a 10-dimensional lattice with $\epsilon = 0.01$ and $L = 0.01$. We stress that the dynamics of this site does not reach the value 1.0, as is shown in the zoom on the right (looking in the data files, we find that the biggest value is ≈0.99). Notice that the dynamics out of $\mathcal{C}$ are very similar to the uncoupled case. Then we can say that the dynamics of each site in the coupled case approach that of the uncoupled one (each site "feels" the presence of its neighbors as a small perturbation), with the exception that the image of the region $\mathcal{C}$ is no longer $x = 1$, but rather another point in its neighborhood.

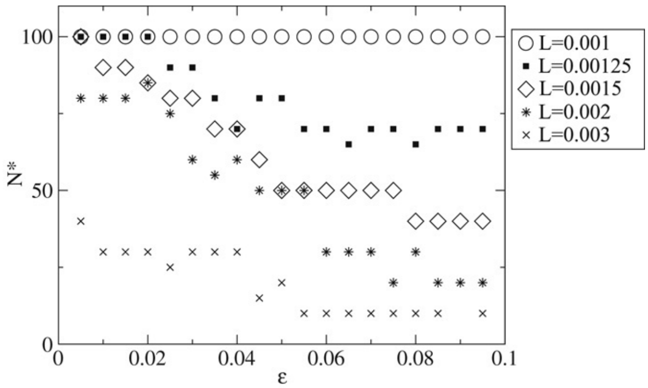


Fig. 3. Critical dimension versus ϵ for different L values — except in the case $L = 10^{-3}$. Remember this is a critical dimension regardless of the dimensions smaller than 10; see text for details.

system dimensions (one for each finite-time Lyapunov). We say we have a fluctuation or oscillation if any one of these distributions crosses zero.

We proceed in the following manner [1]: for each triple (N, ϵ, L), we compute the finite-time Lyapunov distribution and identify whether there is fluctuation; we use a time span equal to 50. For each fixed pair (L, ϵ) there are values of N for which there is oscillation and values for which there is not. The first calculations have shown that for $N = 10$, we have no oscillation when $L < 5 \times 10^{-3}$ and $\epsilon < 0.1$. So we focus our simulations on these values of ϵ, L, and increase N up to a limit where we find oscillating exponents (we do that because, as explained in the introduction, the conjecture concerning CML is that for sufficiently large values of N, we should find UDV). The smaller N value (starting from $N = 10$) for which we find such oscillations is called the critical dimension and is denoted by N^*. So we can plot $\epsilon \times N^*$ for different L values. Such plotting is in Fig. 3.

We remark that the curve in Fig. 3 referring to $L = 10^{-3}$ *does not* present oscillation. In such a case, we have run the simulation up to $N = 100$ and have found no oscillation. We include this curve in the graphics to stress the difference of the behavior of $L = 10^{-3}$ and the other values.

The curves in Fig. 3 strongly suggest that there exists a small region in the parameter space $L \times \epsilon$ near $(0, 0)$ for which there is no UDV. This is so because, as Fig. 3 clearly shows, for fixed L, the critical value N^* increases when we decrease ϵ and, reciprocally, for fixed ϵ we have that N^* increases while we decrease L.

So we fix ourselves on L values no greater than 10^{-3}. As an example, see Fig. 4. It shows the number P of positive finite-time Lyapunov exponents versus time for different lattice sizes. Notice that for these very small ϵ and L values,

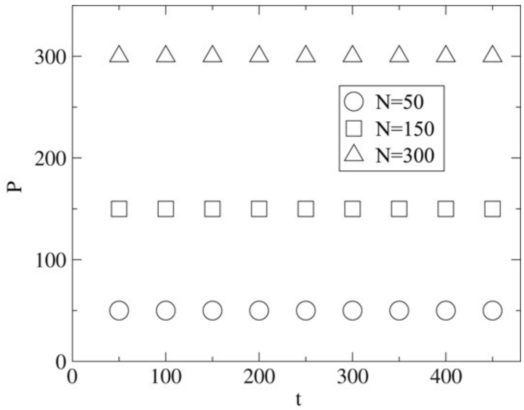
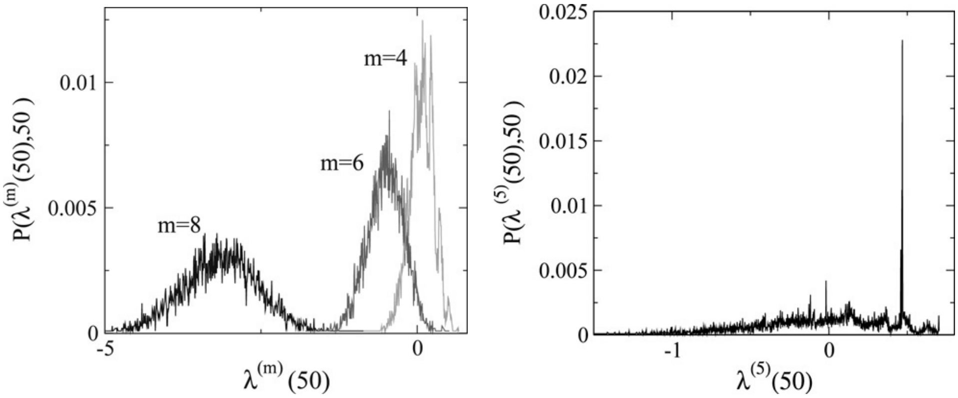


Fig. 4. Number of positive Lyapunov exponents for different N values when $L = \epsilon = 10^{-3}$.



(a) Distributions of the mth exponent for $N = 8$, $L = 10^{-3}$, $\epsilon = 0.1$. Notice that the distributions move away from the distribution of the smallest exponent.

(b) Smallest exponent distribution for $N = 5$, $L = \epsilon = 10^{-3}$. Notice that the distribution is not Gaussian and crosses zero, indicating the existence of UDV.

Fig. 5. Finite-time Lyapunov distributions. We used a time span equal to 50.

the graphic shows that the number of positive finite-time Lyapunov exponents does not vary, indicating the absence of UDV in such cases.

Such results are rather exciting, since in the literature we usually find UDV for lower N values [15]. Although the dimension has never been explored as a parameter, as we did in this work, normally it is not necessary to use N values as big as 150 or 300 to find UDV. However, we cannot keep on with this method since, in fact, we are not allowed to extrapolate our results because they are *negatives*, i.e., there is no evidence of UDV on them, but this does not mean evidence of no UDV. From this point on, we use the values $\epsilon = L = 10^{-3}$ as our starting point to look for the absence of UDV.

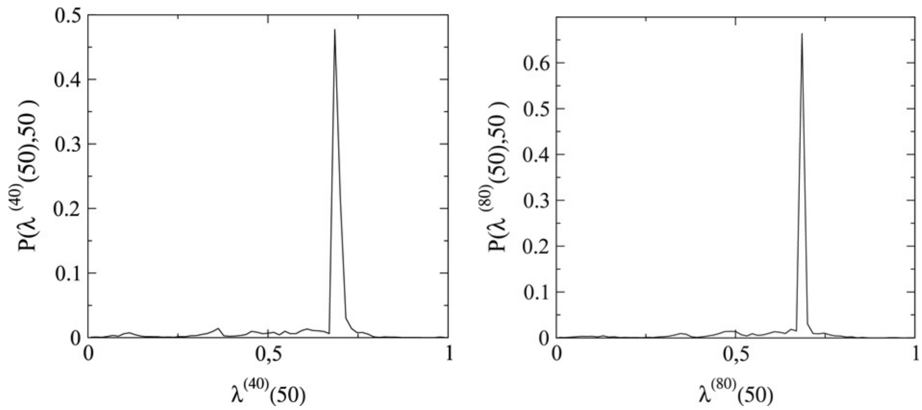
To extrapolate our results, we use the following. We measure the finite-time Lyapunov distributions for different N values and $L = \epsilon = 10^{-3}$. We obtain the mean and the variance of each one of the distributions and investigate how they scale with N. Fig. 5(a) is an example of how the distributions for increasing exponents depart from each other. The mean moves away from the smallest one (which is quite natural, since the exponents are given in increasing order) and the variances become narrower. Therefore we can focus only on the smallest exponent. That is, if a distribution for the smallest exponent does not cross zero, the same holds for the distributions of the greater exponents (see Fig. 5(a) again).

Table 1 shows the mean $\langle\lambda^{(N)}(50)\rangle$ and the variance σ of the smallest exponent distributions for a time span 50 and for different N. Two such distributions are shown in Fig. 6; as a matter of comparison, we also show in Fig. 5(b) the smallest exponent distribution for the same coupling and plateau values and $N = 5$.

Table 1
The mean $\langle \lambda^{(N)}(50) \rangle$ and the variance σ for different N values

N	$\langle \lambda^{(N)}(50) \rangle$	σ	$\langle \lambda^{(N)}(50) \rangle_G$	σ_G
40	0.636445	0.1411238	0.685446	0.0131350
50	0.625132	0.1468947	0.685446	0.0096396
60	0.626850	0.1467468	0.685446	0.0112120
70	0.629224	0.1477443	0.685446	0.0100170
80	0.635342	0.1329479	0.685446	0.0113740

We show also the mean $\langle \lambda^{(N)}(50) \rangle_G$ and the variance σ_G for a Gaussian fitting.



(a) Smallest exponent distribution for $N = 40$. (b) Smallest exponent distribution for $N = 80$.

Fig. 6. Finite-time Lyapunov distributions for $\epsilon = L = 10^{-3}$. We used a time span equal to 50.

From the table, it is clear that the mean is approximately constant around 0.63 and the variance is a non-increasing function of N. Hence, extrapolating these results, we have that the distributions do not cross zero as N is increased to values bigger than 80. This suggests the absence of UDV for all $N \geq 40$ and $L = \epsilon = 10^{-3}$.

In many works concerning shadowing theory, the distributions are Gaussian. Although [12] have shown that this is to be expected only when the variables are completely uncorrelated, the Gaussian approximation is still important when we intend to compare. So we can also try to fit the distributions by a Gaussian. Indeed, our distributions are nearly Gaussian, regardless of some fluctuations on the left of the mean (see Fig. 6). Table 1 shows the mean $\langle \lambda^{(N)}(50) \rangle_G$ and the variance σ_G for the Gaussian fitting. The same results hold: the mean does not approach zero and the variance is a non-increasing function of the dimension.

4. Concluding remarks

The phenomenon of UDV has been rousing the attention of the scientific community due to its rich properties and strong consequences. It seems to be a typical phenomenon of non-hyperbolic high dimensional systems. Because of this, it is to some extent expected that a CML should exhibit UDV when we increase the lattice sizes up to very big values. In this work, we investigated how the presence of UDV depends on the phase space dimension, and we were able to construct an explicit example of a CML with no UDV at high dimensions.

Although our local dynamics are not hyperbolic, we did not show that the global dynamics are also not hyperbolic for non-zero coupling. We discuss this point because it could be conjectured that the non existence of UDV for the high dimensional lattice stems from an emerging hyperbolicity. But we remark that: (i) the dynamics at each site is very similar to the uncoupled case (as noticed in Section 2), so that it is highly probable that the dynamics continues to be non-hyperbolic after the coupling; and (ii) it would be really a strange (and new) result to "create" hyperbolicity through coupling.

The comments of the previous paragraph lead us to some other interesting questions. It is known that hyperbolic systems form an open set (in some adequate topology). So, if we couple some hyperbolic dynamics (such as the so-called Cat Map), it will still be hyperbolic for a very small coupling. Increasing the coupling, we expect to break the hyperbolicity down, which would be detected by the existence of bifurcations. So, we can investigate how the shadowing properties and the (expected) emerging of UDV would behave in such a case. This is the goal of our future work.

Acknowledgement

We are especially thankful to R.L. Viana for the precious communications during the development of this work.

References

[1] We stress that we could not improve our data because they have a high computational cost. To calculate the exponents for large n, we are obliged to deal with matrices $n \times n$. The operations to obtain faithful data when dealing with Lyapunov exponents are quite complex, and the difficulties increase for large matrix sizes. See [7].
[2] H.D. Abarbanel, R. Brown, M.B. Kennel, Variation of Lyapunov exponents on a strange attractor, Journal of Nonlinear Sciences 1 (1991) 175–199.
[3] E. Barreto, P. So, Mechanisms for the development of unstable dimension variability and the breakdown of shadowing in coupled chaotic systems, Physical Review Letters 85 (12) (2000) 2490–2493.
[4] C. Bonatti, L.J. Díaz, Gaël Turcat, Pas de “shadowing lemma” pour les dynamiques partiallement hyperboliques, Comptes Rendus de l’Academie des Sciences Serie 1 330 (7) (2000) 587–592.
[5] E.M. Coven, I. Kan, J. Yorke, Pseudo-orbit shadowing in the family of tent maps, Transactions of the American Mathematical Society 308 (1) (1988) 227–241.
[6] S. Dawson, C. Grebogi, T. Sauer, J. Yorke, Obstructions to shadowing when a Lyapunov expoent fluctuates about zero, Physical Review Letters 73 (14) (1994) 1927–1930.
[7] J.P. Eckmann, D. Ruelle, Ergodic theory of chaos and strange attractors, Reviews of Modern Physics 57 (3) (1985) 617–656.
[8] A. Katok, B. Hassemblat, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, Cambridge, 1995.
[9] Y.C. Lai, D. Lerner, K. Williams, C. Grebogi, Unstable dimension variability in coupled chaotic systems, Physical Review E 60 (5) (1999) 5445–5454.
[10] Y.C. Lai, C. Grebogi, Modeling of coupled chaotic oscillators, Physical Review Letters 82 (24) (1999) 4803–4806.
[11] Ying-Cheng Lai, Unstable dimension variability and complexity in chaotic systems, Physical Review E 59 (4) (1999) 3807–3810.
[12] A. Prasad, R. Ramaswamy, Characteristic distributions for finite-time Lyapunov exponents, Physical Review E 60 (3) (1999) 2761–2766.
[13] T. Sauer, Shadowing breakdown and large errors in dynamical simulations of physical systems, Physical Review E 65 (3) (2002) 036220, 5 p.
[14] T. Sauer, C. Grebogi, J. Yorke, How long do numerical chaotic solutions remain valid? Physical Review Letters 79 (1) (1997) 59–62.
[15] R. Viana, C. Grebogi, S.E.S. Pinto, J.R.R. Barbosa, Pseudo-deterministic chaotic system, International Journal of Bifurcation and Chaos 13 (11) (2003) 3235–3253.
[16] R.L. Viana, C. Grebogi, Unstable dimansion variability and synchronization of chaotic systems, Physical Review E 62 (1) (2000) 462–468.
[17] R.L. Viana, C. Grebogi, S.F. Pinto, S.R. Lopes, A.M. Batista, J. Kurths, Validity of numerical trajectories in the synchronization transition of complex systems, Physical Review E 68 (6) (2003) 067204, 4 p.
[18] R.L. Viana, S.E. Pinto, C. Grebogi, Chaotic bursting at the onset of unstable dimension variability, Physical Review E 66 (4) (2002) 1–9. 046213, 9 p.
[19] G. Yuan, J. Yorke, An open set of maps for which every point is absolutely nonshadowable, Proceedings of the American Mathematical Society 128 (3) (1999) 909–918.