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Dynamics at the interface dividing collective chaotic and synchronized periodic states in a CML

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Abstract

A study is developed focusing the loss of stability of the interface dividing two regions of different spatial patterns on a coupled map lattice using coupling as the parameter guiding the transition. These patterns are constructed over local periodic/chaotic attractors generating regions of synchronized/collective behavior. The discrete feature of the underlying lattice, the anisotropy that stems from such discreteness and its possible change to an isotropic system through coupling with large number of neighbors are also investigated.

Keywords: Coupled map lattices; Extended chaotic dynamical systems; Collective behavior

1. Introduction

During the last three decades we have been dealing with a great increase in researches related to non-equilibrium systems. Words such as "chaos", "dynamical systems", "complex behavior" and "non-equilibrium" have been found more often in literature. Particularly interesting in physics are the systems with large number of

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degrees of freedom. However, methods imported from equilibrium statistical mechanics seem to be insufficient to describe such systems in a number of situations called "out of equilibrium". Often new mathematical tools are developed to deal with them, but a complete "non-equilibrium statistical mechanics" is far from being completely built up.

The powerful computers developed during the last decade enabled us to intensively investigate those systems using numerical methods. Coupled Map Lattices (CML) is a very well-studied case. Since the pioneering works of Kaneko [1], lattices of coupled maps have been used as toy models to investigate universal properties associated to extended non-equilibrium systems.

Different dynamical behaviors can be observed in such lattices, particularly when the coupling produces an average among the nearest neighbors—what is called the *diffusive coupling*. Frequently, the dynamical systems with such coupling exhibit the *non-trivial collective behavior* (NTCB) [2], which is characterized by a well-defined and regular time evolution of lattice *average* quantities—*despite* the local chaos—indicating the existence of a low-dimensional global attractor. The more the lattice size is increased, the sharper these average quantities become.

Among the states which characterize the lattice, different kinds of transition can be observed according to the map parameters, the coupling or the dimension used. The first investigations have shown that one way to reach NTCB is to use a chaotic local map [2]. At very low coupling we expect that each site follows its chaotic route without being significantly perturbed by its neighborhood, what leads to a stationary regime of lattice average quantities. As the coupling is increased a dynamical average behavior diffuses over the lattice resulting in an emerging NTCB characterized by periodic global averages. Such behavior remains up to the highest coupling values and up to the high lattice dimensions [3,4]. Briefly, the works from Chaté and collaborators [5] state that a sufficient condition for NTCB to exist, either in a CML or in a lattice of coupled ordinary differential equations, is the presence of local chaos, even if periodicity windows exist for particular parameter values. In a previous work [6] this problem was explored in the presence of multiple basins of attraction. According to that work, we can find NTCB if the local phase space contains multiple attractors coexisting at finite distances in parameter space. This last situation was investigated deeply by Martins [7], who used a map with a local period-two attractor. In that work one finds a collective behavior emerging from a local periodic attractor surrounded by chaos in the parameter space—i.e., the chaos plays its role indirectly; it is not found in the uncoupled map.

In this work we introduce a local map which presents two basins of attraction of equal size: one in which every initial condition is attracted to a (linearly) stable fixed point and another where the initial conditions converge to an invariant non-periodic set with typical chaotic behavior. In this way we can better investigate the competition between chaotic and periodic states taking into account only the behavior driven by the coupling. Specifically, we are interested in the dynamics at the interface of the lattice that divides two regions of different spatial patterns—collective and synchronized—in as much as a good understanding of the diffusion through the lattice demands an understanding of that dynamics. Each of these regions dividing the lattice presents a behavior associated to one of the two basins of attraction of the local map, that is, the lattice is initialized with half of its sites in one of such basins of attraction and the other half in the other basin.

The way the dynamics evolves depends on three parameters: the first is the coupling strength, called coupling parameter and denoted here by ε. The second is the radius R that determines the neighborhood of each site to be considered in the coupling. The choice of a radius means we shall be using a circular neighborhood. Of course, as the lattice is spatially discrete, the neighborhood is not a perfectly circular one. Such a discrete feature introduces an anisotropy into the lattice, what motivates the third parameter of our investigation: the interface slope θ. Varying the neighborhood size and the slope we shall be able to investigate exactly to what extent the anisotropy is relevant, it is usually assumed that it is not important when the number of neighbors is large. A work in this direction has been done by Rudzick and collaborators [8], but here we focus on the interface transition from rest to some finite speed and on the role of the NTCB. Furthermore, in Ref. [8] the interface had a unique direction and the average was done over a fixed number of neighbors, while here we are varying both parameters and relating them to the observed transitions.

2. The local map

We have chosen a map limited between [0, 1], with only two basins of attraction (length 0.5 each one) and three fixed points: one, linearly unstable, at 0.5 which is in charge of separating the two basins of attraction; another at 0.25, which is linearly stable, and the last, linearly unstable, at 0.75. The fixed point at 0.25 attracts all orbits starting in [0, 0.5) and so it is related to the periodic behavior of the map. The fixed point at 0.75 is embedded in a chaotic invariant set (see below). From now on, we shall refer to the region [0, 0.5) of the local map as the periodic or synchronized region, and (0.5, 1] as the collective one. By an abuse of language, in the same way we shall refer to the lattice region where all the sites have values smaller than 0.5 as the periodic or the synchronized one and that where the site values are bigger than 0.5 as the collective or chaotic region.

As it is well known from the dynamic systems theory, in an one-dimensional map a fixed point is linearly unstable when its derivative has absolute value bigger than one and (linearly) stable when it has absolute value smaller than one. Therefore, dealing with an unstable or stable fixed point is a matter of building up the map with straight lines whose slopes have absolute values higher or smaller than one (of course, it is not necessary to use straight lines, but we take the simpler situation).

The local map, given by

$$x_{t+1} = f(x_t) \tag{1}$$

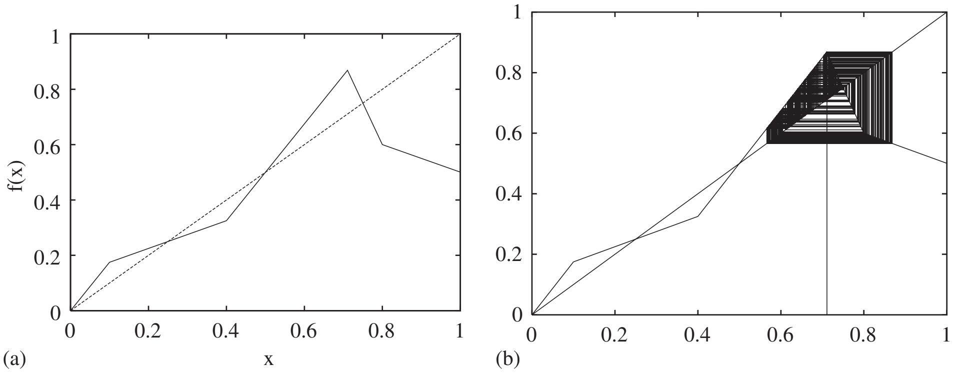


Fig. 1. The local map: (a) the local map; the dashed line is the identity; (b) the first 300 iterations of a dynamics beginning in the invariant set (0.5667,0.8700).

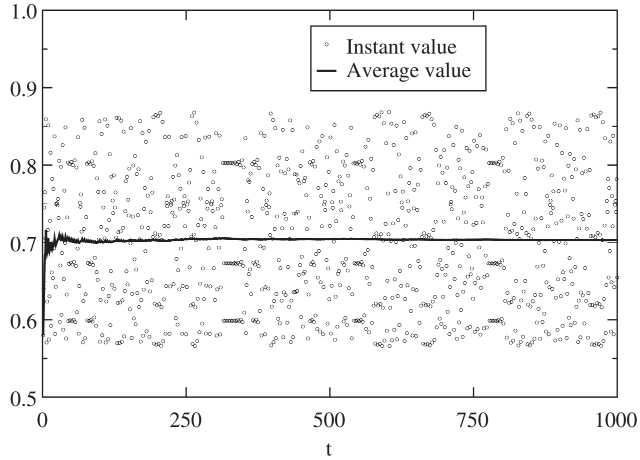


Fig. 2. The local map instant value and its average value versus time.

with

$$f(x) = \begin{cases} 1.750x & (0.000 \leqslant x \leqslant 0.100)\,, \\ 0.500x + 0.125 & (0.100 \leqslant x \leqslant 0.400)\,, \\ 1.750x - 0.375 & (0.400 \leqslant x \leqslant 0.711)\,, \\ -3.000x + 3.00 & (0.711 \leqslant x \leqslant 0.800)\,, \\ -0.497x + 0.998 & (0.800 \leqslant x \leqslant 1.000) \end{cases} \tag{2}$$

is presented in Fig. 1(a). The map has an invariant set $\mathscr{U} = [0.5667, 0.8700]$, in the sense that $f(\mathscr{U}) \subset \mathscr{U}$ (see Fig. 1(b)). Notice that the fixed point at 0.5 is in charge of the division between the basins of attraction.

We plot in Fig. 2 the local map instant value and the local map mean value versus time for an initial condition in region $(0.5, 1.0]$, i.e., the basin of attraction related to the collective behavior. By average value at time t we mean temporal average up to time t. Such a plotting will be important later to deduce properties that arise from the coupling.

3. The coupled map lattice

There are different ways of defining a Coupled Map Lattice (CML). Here we adopt a more restrictive definition that is enough for our purposes. Readers interested in broader concepts of CML may see Refs. [9,10].

The CML is an array of $N \times N$ local maps where each element of the array (called *site*) can interact with the others. Calling $x_t^{i,j}$ the field value at the ith, jth site at time t, with $i,j = 1, 2, \ldots, N$ and $\tilde{x}_{t+1}^{i,j}$ the field value at the ith, jth site after one iteration of the local map, the lattice evolution can be described as follows: (i) first, each site is updated in time through Eq. (1) as

$$\tilde{x}_{t+1}^{i,j} = f(x_t^{i,j}) . \tag{3}$$

(ii) Subsequently, each local map is averaged over a set of neighbors $(x_t^{i',j'})$. So we can write

$$x_{t+1}^{i,j} = (1-\varepsilon)\tilde{x}_{t+1}^{i,j} + \frac{\varepsilon}{S}\sum_{i'j'} \tilde{x}_{t+1}^{i',j'} . \tag{4}$$

We can rewrite (3) and (4) briefly as

$$x_{t+1}^{i,j} = (1-\varepsilon)f(x_t^{i,j}) + \frac{\varepsilon}{S}\sum_{i'j'} f(x_t^{i',j'}) , \tag{5}$$

where ε is the coupling parameter; when $\varepsilon \to 0$ the maps evolve independently, i.e., the sites become decoupled. The term S is called coordination number and indicates how many neighbors we are averaging over.

4. Initial and boundary conditions

The dynamics has been made over the lattice initialized with two regions (collective and synchronized), which means that we have started half of the lattice sites in the basin of attraction chaotic and the other half part in the basin of attraction periodic. The neighborhood is taken as a circular one: given a site, we are averaging over all sites in a distance equal or smaller than R, the radius of the neighborhood. The closest distance between two sites is considered as the unit of distance.

Although periodic boundary conditions are traditionally used in CML, here such conditions would yield at the boundary a discontinuity of the interface which divides the synchronized and the collective regions (except for two particular directions), because we shall explore straight initial interfaces with different orientations (see

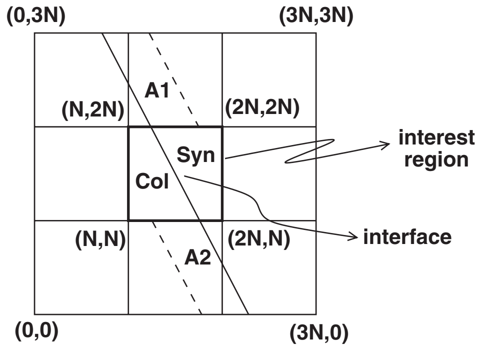


Fig. 3. Lattice schematic representation. The labels Col and Syn indicate the collective and synchronized regions. The dotted lines indicate how the interface would have been if we had used periodic boundary conditions. Notice that A1 is a synchronized region and A2 is a collective one; whereas if periodic boundary conditions were adopted, A1 would be collective and A2 synchronized—exactly the opposite.

Fig. 3). So, to avoid this problem and also the boundary effects we constructed a lattice three times bigger than the size over which we measured, i.e., it has $3N \times 3N$ sites with all the measurements of interest carried out on the square with corners at (N, N), $(N, 2N)$, $(2N, N)$, $(2N, 2N)$ (see Fig. 3).

Let us call this region where the measurements are performed as the "interest region". In this work, when we refer to the lattice size we shall be always talking about the size of the interest region (see Fig. 3 again). Of course, sites out of the interest region must go on evolving as described by Eqs. (3) and (4) but when we approximate the corners we simply use a neighborhood as large as possible to fall into the lattice. This approach is not expected to perturb the interface dynamics on the interest region since the correlation lengths are smaller in regions with small coordination numbers.

We investigate the transition from an interface at rest to a moving one as a function of the coupling parameter—looking for its critical value in which the transition begins, and how it scales with the radius R and the slope θ. This slope is measured from the "vertical" axis (i.e., the axis from (0,0) to $(0, 3N)$ in Fig. 3). Notice that the radius R determines the coordination number S.

5. The NTCB

Before going ahead, we should assure that our lattice has in fact the correct dynamical behavior in both regions. To be sure of the presence of a NTCB, we measure the instant average value versus time (for a 300×300 lattice and 6000 iterates) with all the sites initialized in the collective region. Some 500 initial iterates are rejected as transients. We present the results for a typical time window in Fig. 4, for different coupling values, $\varepsilon = 0.05, 0.3, 0.4$ and 1.0. The lattice average increases roughly with increasing coupling values, the plots

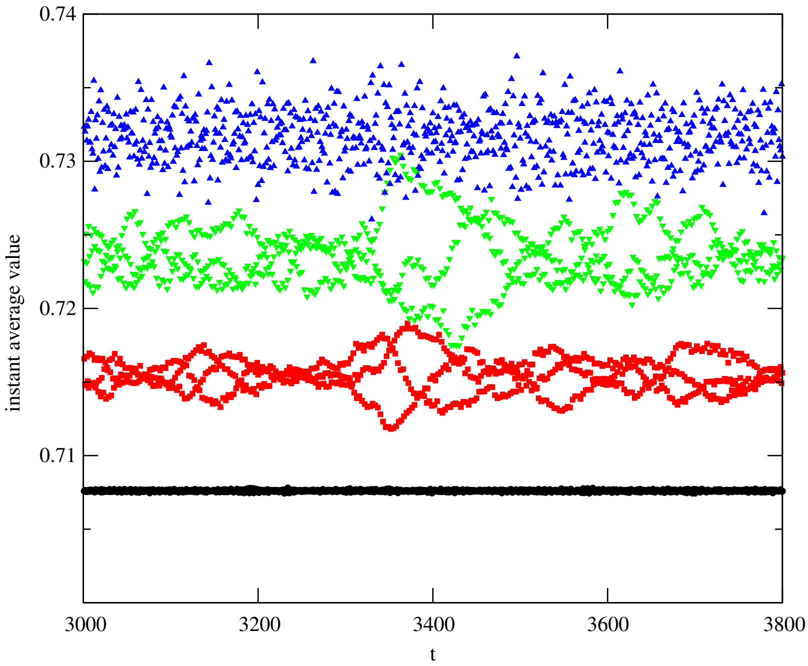


Fig. 4. Details of the time evolution of the average lattice value for different coupling parameters when all sites are in the chaotic region. From bottom to top, values of $\varepsilon = 0.05, 0.3, 0.4$ and 1.0 are shown. In this figure $R = 1$. See text for details.

for these different ε values appear from bottom to top. At very low coupling the sites are nearly independent and a constant value very close to the time average of the uncoupled system (Fig. 2) is observed for the instant average over the lattice. As the coupling increases an intermittency between a quasi-periodic cycle three and a period one appears ($\varepsilon = 0.3, 0.4$). For couplings above these values, the quasi-periodic and the period one cycles gradually merge into a period one with large fluctuations ($\varepsilon = 1.0$). For clarity we did not include intermediate values of ε in Fig. 4.

Comparing Figs. 4 and 2 we observe that the average value in the coupled system oscillates in time, according to the descriptions of NTCB [3]. For the region with the periodic attractor, it is quite obvious that if we initialize all sites in the lattice with values smaller than 0.5 they will rapidly evolve to a synchronized state with value 0.25 (the value of the attracting fixed point).

It can be argued that when both regions are coupled, this construction will not remain. In fact, above a threshold value of the coupling, it would not. But, below this transition value, the average field values at the interface and its fluctuations on the chaotic side are very close to the values deep inside the NTCB region. This happens even for the largest coordination number used ($R = 5$), as is shown in the histograms of Figs. 5(a) and (b) for $\varepsilon = 0.05$ and $\varepsilon = 0.3$, respectively.

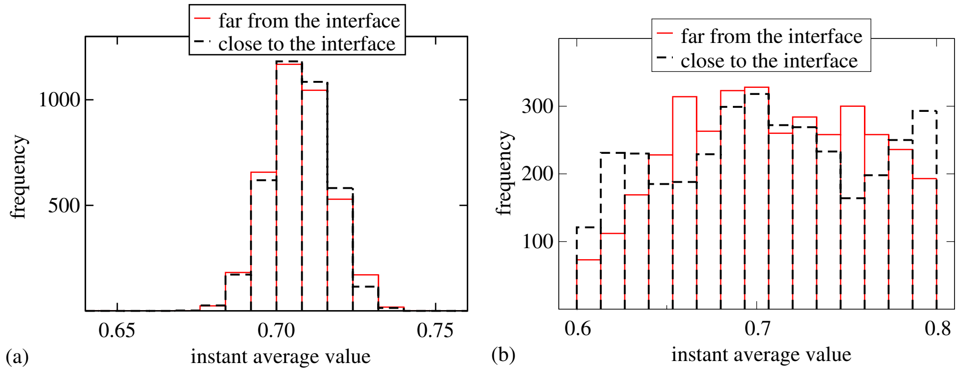


Fig. 5. Histograms of field values close and far from the interface for $R = 5$ (a) $\varepsilon = 0.05$, (b) $\varepsilon = 0.3$.

6. Measurements and discussion

To investigate the interface evolution we measure the dependence of its velocity on the control parameters. The velocity is conveniently defined as

$$v = \frac{N^2}{2T}\,, \tag{6}$$

where T is the time (number of iterates) the interface takes to pass over all periodic sites, i.e., the time demanded by all periodic sites to be infected by the collective ones (we should also consider the possibility of the collective sites to be infected by the synchronized ones, but we know a posteriori that such a possibility does not happen and below we will explain why). Obviously, if there is no transition the velocity equals zero, for in this case $T \to \infty$. $N^2/2$ is the number of sites initially in the synchronized region (which is equal to the number of sites initially in the collective region). We see then that the velocity is given in ‘‘number of sites infected per unit of time’’. The use of such a unit to measure velocity enables us to compare velocities in lattices of different sizes.

Recalling, we define the transition as change from a lattice state in which the interface does not move to another state with moving interface. The first situation we call ‘‘equilibrium’’, whereas the other we name by ‘‘non-equilibrium’’. A non-zero velocity indicates that a transition has happened. Beyond detecting the existence of a transition, the measurement of the velocity allows us to give a more detailed characterization of it in the sense that we shall have a quantitative order parameter to make statements about the dynamics.

We have varied the coupling parameter ε, the slope θ and the radius R and identified the values for which the transition occurs. We use a lattice of 300×300 sites (with this size we find good convergence of the average quantities). When the radius is small (corresponding to the classical situations of first neighbors coupling) there is a strong dependence of the transition on the parameter θ. This was, to some extent, expected since the lattice has a natural anisotropy that stems from its discrete nature. For second neighbors coupling, for example, when the slope is $\approx\pi/4$ a periodic site near interface is able to interact with a collective one, which pushes its value away from 0.25. On the other hand, this situation is forbidden when $\theta \approx 0$. Both situations are illustrated in Fig. 6.

We have found that the transition occurs faster for increasing coupling parameter and increasing radius values. Figs. 8(a) and (b) show that the transition does not occur for any coupling value smaller than 0.4. Before analyzing these graphics in more detail we want to explain how the transition occurs.

We can ask what kind of behavior is induced on the interface by the two regions (collective and synchronized). This is investigated by measuring the interface average value and its dependence on the parameters θ, ε and R. We vary such parameters only for those values for which the transition has not yet happened. This is quite reasonable: we want to know only how the dynamics evolves from an equilibrium situation to an out of equilibrium one, which means that we want to know how the interface value increases up to a value where it starts moving. Moreover, from the computational point of view it would be quite complex to compute values in a moving interface. As the transition happens around $\varepsilon = 0.4$, we restrict ourselves to smaller coupling values. The results after 6000 iterations are shown in Figs. 7(a) and (b).

In Figs. 7(a) and (b) we clearly see that the interface mean value increases with increasing coupling and with increasing coordination number. Now we can understand how the transition arises: for small values of ε and R, each time a site on the interface is averaged over a set of neighbors its value does not change enough to produce a motion of the whole interface. When the parameters are increased, the values at the interface become higher. This can be caused only by an increase in the values of the sites in the collective region since the instantaneous average over the chaotic region increases with the coupling (Fig. 4) [11] and the coupling intensity has no effect on the periodic region far from the interface. So there are critical values of the coupling strength and radius for which the interface values become collective

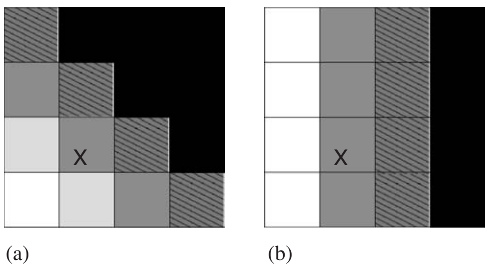
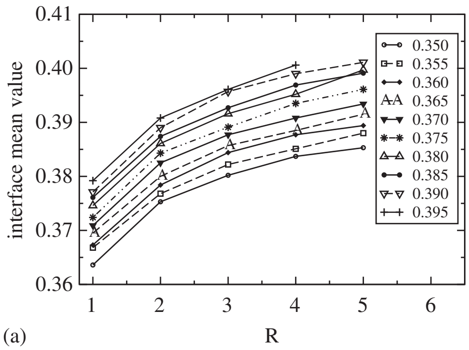


Fig. 6. Second neighbor coupling near interface. The closer periodic sites are to interface the more, their values increase since they interact with sites with higher values (the collective ones). We may represent that schematically as if the lattice was composed by layers whose sites have values increasing in the direction of the interface (indicated by a gray-scale). The site marked with an "X" can interact with the sites of the collective region (the black ones) when $\theta \approx \pi/4$ (a), whereas it cannot when $\theta \approx 0$ (b).



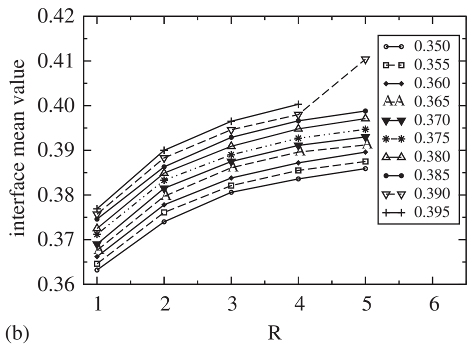


Fig. 7. Interface mean value. For $R = 5$, there is no corresponding point on the curves ε because for such radii and coupling values the transition has already happened: (a) interface average value for different coupling values and $\theta = 0$; (b) interface average value for different coupling values and $\theta = \pi/4$.

ones, the fluctuations coming from the chaotic region are not dumped any more moving the interface to the synchronized region. But it should be remarked that, these values are quite below the average between the periodic mean value and the collective one (the first equals to 0.25 and the other ranging from ≈0.71 up to ≈0.73 for ε; see Fig. 4), showing that the transition does not happen according to a simple mean field interpolation. This is expected since we are dealing with a diffusive coupling which essentially averages over the neighborhood. In fact, in this line of reasoning, the local chaotic fluctuations, characteristic of the NTCB, and the increase—with the coupling—of the average lattice values in the chaotic region have an essential role in capturing the synchronized sites to the collective region.

Figs. 8(a) and (b) show the velocity versus radius for different values of ε, θ, after 6000 iterations. The different curves represent different coupling values near the transition. The coupling increases ($0.42 < \varepsilon < 0.49$) from bottom to top in these figures. We have found out that a (strong) dependence on the slope θ remains even at high neighborhoods such as those determined by radii of length 4 and 5—what corresponds to neighborhoods of 48 and 80 sites, respectively. The measurements using the 300×300 lattice have shown that the transition starts at $R = 2$ for null interface slope, but for slopes near $\pi/4$ the interface starts moving when $R = 1$. In Figs. 8(a) and (b) we also see that even for large coordination numbers (such as those determined by radii values $R = 4$ and $R = 5$) the dependency on the parameter θ strongly persists. For example, when $R = 4$ the transition starts near $\varepsilon = 0.44$ for $\theta = 0$ (first non-null velocity in Fig. 8(a) corresponding to the abscissa $R = 4$) and near $\varepsilon = 0.42$ when $\theta = \pi/4$ (first non-null velocity in Fig. 8(b) corresponding to the abscissa $R = 4$). However, notwithstanding the transition starting first for $\theta = \pi/4$, the interface velocity is higher (when it is non-null) for $\theta = 0$, as can be seen by comparing the non-null values of velocity in Figs. 8(a) and (b). That is, the "harder" direction also propagates information faster.

There is another important difference between the cases $\theta = 0$ and $\theta = \pi/4$. For $\theta = 0$ the dependency of the velocity on the radius is approximately linear with the

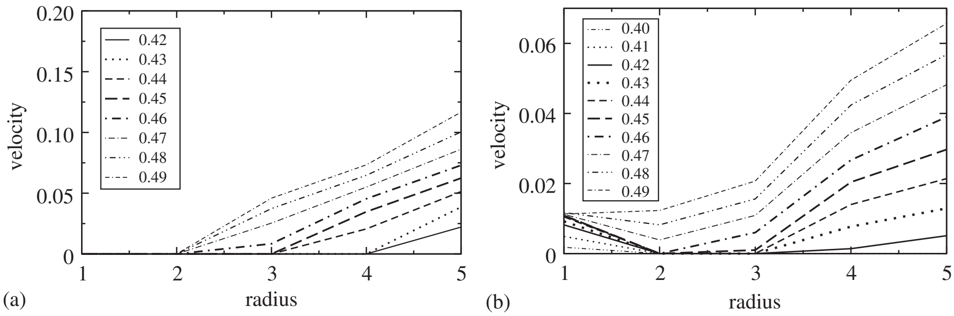


Fig. 8. Velocity versus radius for different coupling parameters and radii. Each curve represents a different coupling parameter ε. The coupling increases ($0.42 < \varepsilon < 0.49$) from bottom to top in the different curves of both figures. (a) $\theta = 0$, (b) $\theta = \pi/4$. For each curve there are points corresponding to integer radii values, i.e., from $R = 1$ up to $R = 5$.

radius (Fig. 8(a)) while for $\theta = \pi/4$ there is a minimum at $R = 3$ and $R = 4$ (Fig. 8(b)). A tentative explanation is to attribute this difference to the asymmetry in the neighborhood, since with first neighbors coupling we do not count the diagonal sites and with second and third neighbors the diagonal sites counted are the same.

7. Conclusions

In this work we have explored the evolution of the interface separating a collective region and a periodic one in a two-dimensional lattice of coupled maps with diffusive coupling. Throughout the paper we have focused mainly on the transition where the interface starts moving and on its dependency on the coupling parameter, the coordination number and the initial orientation of the interface.

Regarding the transition we notice that the periodic sites are correctly described by their local asymptotic values. But, in the chaotic region, the correlation introduced by the coupling results in an increase in the average values of these sites. More than that, the instantaneous average values over the lattice fluctuate in time—presenting the characteristic low-dimensional global attractor [12]—and the local chaotic sites may have values well above these average fluctuations. These local fluctuations combined with the global average increase are responsible for capturing sites from the periodic region to the chaotic one, resulting in the interface motion.

In an attempt to establish isotropy on the lattice, we have used a large number of neighbors (up to 80 sites in a circular neighborhood) in the coupling but this has not eliminated the dependency of the critical coupling on the orientation of the interface. We have also noticed that after the transition the interface growth was smooth, as would be expected in a KPZ like system [13] with a null gradient modulus coupling term [13,8].

Based on the above results, we claim that, due to the underlying lattice anisotropy, CML should be used with care to model continuous-extended systems. In other words, it seems that the continuous limit of a spatial pattern represented by a CML, which rigorously speaking should be reached from the limit of site distances tending to zero, is not well approximated by a CML with large coordination numbers.

Acknowledgements

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